SIAM J. NUMER. ANAL.
Vol. 32, No. 3, pp. 924-951, June 1995

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NYSTRÖM'S METHOD AND ITERATIVE SOLVERS FOR THE SOLUTION OF THE DOUBLE-LAYER POTENTIAL EQUATION OVER POLYHEDRAL BOUNDARIES *

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Abstract. In this paper we consider a quadrature method for the solution of the double-layer potential equation corresponding to Laplace's equation in a three-dimensional polyhedron. We prove the stability for our method in the case of special triangulations over the boundary of the polyhedron. For the solution of the corresponding system of linear equations, we consider a two-grid iteration and a further simple iteration procedure. Finally, we establish the rates of convergence and complexity and discuss the effect of mesh refinement near the corners and edges of the polyhedron.

Key words. potential equation, Nyström's method, two-grid iteration

AMS subject classifications. 45L10, 65R20, 65F10

1. Introduction. One popular method for solving boundary value problems for elliptic differential equations involves the reduction to boundary integral equations. For instance, the Dirichlet problem for Laplace's equation in a bounded and simply connected polyhedron $\Omega \subseteq \mathbf{R}^3$ or the Neumann problem for the same equation on $\mathbf{R}^3 \setminus \Omega$ can be reduced to the second-kind integral equation $Ax = y$ over the boundary $S := \partial\Omega$ (cf., e.g., [19]), where $A = I + 2W$ and

$$Wx(Q) := [1/2 - d_\Omega(Q)]x(Q) + \frac{1}{4\pi}\int_S \frac{n_P \cdot (Q-P)}{|P-Q|^3} x(P) d_P S,$$

$$d_\Omega(Q) := \lim_{\epsilon \to \infty} \frac{\mu(\{P \in \Omega : |P-Q| < \epsilon\})}{\mu(\{P \in \mathbf{R}^3 : |P-Q| < \epsilon\})}. \tag{1.1}$$

Here n_P denotes the unit vector of the interior normal to Ω at P and $\mu(Z)$ is the Lebesgue measure of Z. Note that, since the boundary S is not smooth, W is not compact. The kernel function $k(Q,P) := \frac{1}{4\pi} n_P \cdot (Q-P)|P-Q|^{-3}$ vanishes if P and Q lie on the same face of S. However, if P and Q tend to an edge point of S and lie on different faces, then $k(Q,P)$ is of order $|P-Q|^{-2}$. Thus the kernel function of W has a fixed singularity at the set of edge points.

For the numerical solution of $Ax = y$, various methods have been introduced. The first method was the so-called panel method, i.e., piecewise constant collocation (cf. [30], [13], [31], [2]). This method has been proved to converge if Ω satisfies the condition introduced by Wendland in [30] (cf. (3.2)). Moreover, Kral and Wendland [16] (cf. also [17], [1]) have shown that the panel method is stable for the case of certain rectangular domains Ω. Arbitrary polyhedral domains have been considered in [23]. Elschner [9] has analysed the Galerkin method with piecewise polynomial trial functions over arbitrary polyhedrons, and the Galerkin method together with an approximation of the Lipschitz boundary by smooth surfaces has been investigated by Dahlberg and Verchota [7]. In [21] a quadrature method that is similar to the methods of Chandler and Graham [6], Kress [18], and Elschner [8] for the corresponding

* Received by the editors May 1, 1993; accepted for publication (in revised form) November 26, 1993.

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equation over polygonal boundaries has been considered. The advantage of quadrature methods lies in the full discretization of the equation. For other discretization schemes, a further discretization step is needed in order to compute the entries of the stiffness matrix. The analysis of this step, however, is still incomplete. The first analysis of an iterative solution is due to Wendland [30], who has considered a discretized version of Neumann's iteration. Multigrid methods for the solution of the arising linear systems have been analysed by Schippers [26], [27], Atkinson and Graham [4], [5], and the author [20] for the one-dimensional case as well as by Hebeker [12] and Atkinson [3] for the two-dimensional case (cf. also [11], [10]). The main point in the convergence proof is, roughly speaking, the following: split the discretized operator into the sum of two parts, where the first part is the discretized double-layer operator restricted to a neighbourhood of the set of edge points. After multiplying the whole discretized equation by the inverse of this first part, a second-kind equation with discretized compact operator arises. For this situation, the well-known theory of multigrid methods applies. The drawback of these methods, however, is that the first part of the discretized operator is to be inverted. Since this step means the inversion of a large matrix, no improvement of the asymptotic order of complexity can be achieved. Therefore, the authors of [27], [3] recommend better variants of multigrid procedures for which no convergence proofs are yet known.

The aim of the present paper is to analyse a simple quadrature method that was mentioned in [21] without proof. In comparison with the methods investigated in [21] it is much easier since it is based on a different type of the so-called method of singularity subtraction. More exactly, in [21] the integral $\int k(Q,P)x(P)dP$ has been written as $\int k(Q,P)[x(P)-x(R)]dP + x(R)\int k(Q,P)dP$, and the quadrature rule has been applied to $\int k(Q,P)[x(P)-x(R)]dP$. The point R has been chosen from the set of edge or corner points in an appropriate way depending on Q. In the quadrature method of the present paper we easily take $R = Q$. This choice enables us to prove stability without the so-called localization technique and to deduce the convergence of a simple two-grid iteration scheme analogously to the one-dimensional case (cf. [5], [20]). We remark that our subtraction technique may be advantageous also if additional terms with discontinuities along the diagonal are added to the kernel function. These additional terms arise if one considers domains with curved boundaries or boundary integral equations corresponding to the Helmholtz equation. In detail we shall describe the Nyström method and its iterative solution in §2. The first iteration scheme is just the classical two-grid method (cf. [11], [10]), where Jacobi's iteration is used for the smoothing step and Nyström's interpolation for the restriction and prolongation. However, when applied to our double-layer operator, it shows the same new features as detected in the case of one-dimensional boundaries (cf. §4 and [4], [5], [20]). Namely, the convergence ratio of the iteration process does not tend to zero if the mesh size of the fine and coarse grid tends to zero. Rather, the convergence ratio depends on another property of the grids (cf. (3.3) and (4.2)). In particular, the ratio can be improved by cutting off a larger neighbourhood of the set of edge points (i.e., by choosing a strip with a larger ϵ in (2.2), where ϵ is still of the order of the mesh size). Besides the two-grid iteration for the solution of the arising linear system of equations, we also define an iteration scheme that corresponds to the Neumann series expansion. This iteration was first introduced by Wendland [30] for the case of the panel method. In §3 we prove the stability of the Nyström method. It turns out that our quadrature method is stable if a certain finite-section procedure (cf. (2.2)) is convergent. For instance, if Wendland's condition (cf. [30] or (3.2)) is satisfied or if the

corners of Ω are rectangular and graphs of Lipschitz functions, then this finite-section procedure is convergent and our Nyström method is stable. We note that the quadrature method in [21] is based on a different type of finite-section technique, and its stability can be proved also for rectangular non-Lipschitz corners. The convergence of the iteration schemes will be analysed in §4. We shall show that the two-grid iteration is convergent if the Nyström method is stable. The Neumann iteration converges if Wendland's condition (cf. [30] or (3.2)) is satisfied. In §5 we shall introduce special nonuniform triangulations and obtain the same asymptotic error estimates as in [21]. From these estimates we deduce the complexity,[1] i.e., the number of operations necessary to compute the approximate solution with an error less than a prescribed small number. If N is the number of linear equations of the Nyström method, then one needs $O(N^3)$ operations to solve the system of equations by Gaussian elimination. This order can be reduced to $O(N^2)$ if one applies the iteration scheme corresponding to the Neumann series expansion. However, for domains Ω with complicated geometry, $O(N^2)$ means an estimate by constant times N^2, where the constant may be large. The application of the two-grid iteration over quasi-uniform grids provides an order $O(N^2)$ with a smaller constant. Unfortunately, for highly graded meshes, the two-grid method has the same asymptotic order of complexity as Gaussian elimination, and the number of necessary operations reduces by a constant factor, only. In other words, comparing the Nyström method over an "optimally" graded mesh solved by Gaussian elimination with the Nyström method over an almost uniform grid solved by two-grid iteration, we get the same asymptotic order of complexity. However, we expect the two-grid method to have a smaller constant in the complexity estimate. Comparing the two-grid method and Gaussian elimination for the same mesh-grading parameter, the complexity of the two-grid iteration is less if N is sufficiently large. Finally, in the last section we present numerical experiments in order to confirm our theoretical results. In particular, we compare our iterations with the Gaussian algorithm, the GMRES [25], [29], [14], [28], and the GMRES with two-grid preconditioner. Note that throughout the paper we have chosen the composite midpoint rule as the quadrature formula. Higher-order rules can be treated similarly (cf. [21]), but they do not improve the asymptotic order of convergence.

2. The quadrature method and the iteration procedures.

2.1. Derivation of the Nyström method. Let us start with the singularity subtraction. Taking into account that the constant function is an eigenfunction of W corresponding to the eigenvalue $\frac{1}{2}$ (cf. [19, §1.3]), the equation $Ax = y$ may be written as

$$2x(Q) + \frac{1}{2\pi} \int_S \frac{n_P \cdot (Q-P)}{|Q-P|^3} [x(P) - x(Q)] d_P S = y(Q), \quad Q \in S. \tag{2.1}$$

Now the next step in the discretization of (2.1) is the finite-section method. We take a small strip $Str \subset S$ of width $\epsilon > 0$ around the edges and set $S_\epsilon = S \setminus Str$. We suppose that ϵ is of the same size as the diameters of the subdomains in the partition

[1] Since we shall discuss iterative algorithms for linear systems where the size depends on the mesh refinement, the convergence order depending on the number of degrees of freedom is not sufficient to measure the quality of our methods.

used for the quadrature rule. Instead of (2.1) we consider[2]

$$2x_\epsilon(Q) + \frac{1}{2\pi}\int_{S_\epsilon} \frac{n_P \cdot (Q-P)}{|Q-P|^3}[x_\epsilon(P) - x_\epsilon(Q)]d_P S = y(Q),\ Q \in S. \tag{2.2}$$

Now we take a triangulation $S_\epsilon = \bigcup_{i=1}^N S^i$ of S_ϵ. We denote the centroid of the triangle S^i by P^i and the surface measure of S^i by $\mu(S^i)$. In order to get a quadrature method for (2.2) we replace the integral by a quadrature rule. Thus the Nyström method consists of solving

$$2x_N(Q) + \frac{1}{2\pi}\sum_{i=1}^N \frac{n_{P^i} \cdot (Q-P^i)}{|Q-P^i|^3}[x_N(P^i) - x_N(Q)]\mu(S^i) = y(Q),\ Q \in S. \tag{2.3}$$

As is well known, the solution of (2.3) can be obtained in two steps. First one has to solve

$$2x_N(P^j) + \frac{1}{2\pi}\sum_{i=1}^N \frac{n_{P^i} \cdot (P^j-P^i)}{|P^j-P^i|^3}[x_N(P^i) - x_N(P^j)]\mu(S^i) = y(P^j), \quad j = 1,\dots,N, \tag{2.4}$$

where, taking into account that $n_{P^i} \cdot (P^j - P^i) = 0$ for P^j and P^i on the same face of S, we have set $n_{P^j} \cdot (P^j - P^j)|P^j - P^j|^{-3} := 0$. Using the solution $x_N(P^j)$ of (2.4), the solution x_N of (2.3) is given by Nyström's interpolation

$$x_N(Q) = \frac{y(Q) - \frac{1}{2\pi}\sum_{i=1}^N \frac{n_{P^i}\cdot(Q-P^i)}{|Q-P^i|^3} x_N(P^i)\mu(S^i)}{2 - \frac{1}{2\pi}\sum_{i=1}^N \frac{n_{P^i}\cdot(Q-P^i)}{|Q-P^i|^3}\mu(S^i)},\ Q \in S. \tag{2.5}$$

The denominator in (2.5) is different from 0 (cf. the beginning of the proof of Theorem 3.1).

2.2. The two-grid method. Now we need some notation in order to introduce the two-grid method. Let us follow [11] and denote the operators on the left-hand side of (2.2) and (2.3) by A_ϵ and A_N, respectively. Further, let χ stand for the characteristic function of S_ϵ and χI for the operator of multiplication by χ. Then A_ϵ and A_N take the form (cf. (1.1), (2.2), and (2.3))

$$A_\epsilon = 2[1 - W(\chi)]I + 2W\chi I,\ \ A_N = 2[1 - W_N(\chi)]I + 2W_N\chi I, \tag{2.6}$$

where

$$W_N(\chi x)(Q) := \frac{1}{4\pi}\sum_{i=1}^N \frac{n_{P^i} \cdot (Q-P^i)}{|Q-P^i|^3}\mu(S^i)x(P^i) \tag{2.7}$$

and $[1 - W_N(\chi)]I$ as well as $[1 - W(\chi)]I$ stand for the operators of multiplication by the functions $[1 - W_N(\chi\,1)]$ and $[1 - W(\chi\,1)]$, respectively. Besides the triangulation

[2] Note that the transition to this finite-section equation will play an important role for the stability proof in §3. Since we truncate the boundary before dividing it into subdomains, we get a smaller number of linear equations. The additional approximation error is of the same asymptotic order as the discretization error without finite section. However, numerical tests show stability and smaller errors for the Nyström method without the finite-section step.

$S_\epsilon = \bigcup_{i=1}^{N} S^i$ we consider a coarse grid $S_\epsilon = \bigcup_{i=1}^{N_c} S_c^i$. Let P_c^i stand for the midpoint of S_c^i. We denote the corresponding discretized double-layer operator $W_N \chi I$ over the coarse grid by $W_{N_c} \chi I$. Our aim is to solve $A_N x_N = y$ by a two-grid iteration process. Therefore, we start with an arbitrary initial function x^0 (e.g., $x^0 = 0$) and compute the approximate solutions x^i for x_N by iteration. Suppose we have obtained x^{i-1}. Then x^i will be determined as follows (cf. [11]): We start with the smoothing step

$$x' := \frac{1}{2}[1 - W_N(\chi)]^{-1}\{y - 2W_N(\chi x^{i-1})\} \tag{2.8}$$

and define the residual (defect)

$$d := y - 2[1 - W_N(\chi)]x' - 2W_N(\chi x'). \tag{2.9}$$

The correction term c is the solution of the coarse-grid equation[3]

$$A_{N_c} c := 2[1 - W_N(\chi)]c + 2W_{N_c}(\chi c) = d. \tag{2.10}$$

Finally, the approximate solution x^i of the ith iteration step is given by

$$x^i := x' + c. \tag{2.11}$$

Let us note that there exists a faster algorithm for the computation of x^i due to Atkinson (cf. [5]). Moreover, the algorithm becomes still faster if Nyström's interpolation in the prolongation of the coarse-grid data to the fine grid is replaced by piecewise constant interpolation. We shall describe the details in item (d) of §6.3.

In §4 we shall show that $x^i \to x_N$ and $\|x^i - x_N\| \leq q^i \|x^0 - x_N\|$ holds with $0 < q < 1$, where q depends only on the parameter ϵ_p appearing in the stability conditions for the quadrature method (cf. (3.3), (4.2)). Moreover, we derive the orders of complexity in §5.2. Finally, we remark that there is a possibility of avoiding the finite-section step (2.2) without losing the convergence estimates of §4.1. Instead of throwing away the strip *Str* we may choose the coarse grid equal to the fine grid over *Str* (cf. the first remark following Theorem 4.1). In the case of one-dimensional boundaries this iteration is analysed in [20].

2.3. The Neumann iteration. Since a coarse grid is used in (2.10), the computational work for the two-grid iteration is less than that for a direct method of solution of (2.4). This will be satisfactory if the finer triangulation is nearly uniform (cf. §5.2). However, if we take into account the singular behaviour of the solution $x = (I + 2W)^{-1} y$ and that of the kernel function k corresponding to the integral operator W, then we have to use a strongly nonuniform mesh. In this case the stability condition for the triangulation (cf. (3.3), (4.2)) implies that, from the point of view of asymptotic order, the number of triangles in the coarse grid is nearly equal to the number of triangles of the fine grid. Therefore, we consider another iteration process for the solution of (2.4). This iteration was proposed by Wendland [30] for the case of piecewise constant collocation and is nothing else than a discretized version of Neumann's iteration. We start with $x^0 = 0$, choose a parameter $\kappa > 1$ (cf. the remarks after Theorem 4.2), and write the equation $A_N x_N = y$ in the

[3] Note that we could have also set $A_{N_c} := 2[1 - W_{N_c}(\chi)]I + 2W_{N_c}\chi I$.

equivalent form $\{I + \frac{1}{\kappa}(A_N - \kappa I)\}x_N = \frac{1}{\kappa}y$. The iterative solutions are defined by $x^i := \frac{1}{\kappa}y - \frac{1}{\kappa}(A_N - \kappa I)x^{i-1}$, i.e.,

$$x^i := x^{i-1} + \frac{1}{\kappa}\left\{y - 2[1 - W_N(\chi)]x^{i-1} - 2W_N(\chi x^{i-1})\right\}, \quad i = 1, 2, \ldots . \tag{2.12}$$

In §4.2 we shall prove that $x^i \to x_N$ and $\|x^i - x_N\| \leq Cq^i\|x^0 - x_N\|$ holds for some $0 < q < 1$, where q depends on the geometry of Ω. Note that the two-grid iteration may be preferable if q is close to 1.

3. The stability of method (2.3). Let us consider the space $C(S)$ of continuous functions over S supplied with the supremum norm $\|\cdot\|$. By $\|\cdot\|$ we also denote the operator norm on $C(S)$. We call (2.3) stable if A_N is invertible and $\|A_N^{-1}\| \leq C$ = constant with C independent of the partition $S_\epsilon = \bigcup S^i$. Let us introduce the constant

$$C_1 := \sup_{Q \in S} \frac{1}{4\pi} \int_S \frac{|n_P \cdot (P - Q)|}{|P - Q|^3} d_P S < \infty.$$

By C let us denote a generic positive constant that varies from instance to instance. For fixed i, let E^i denote the union of all faces on S that do not contain the triangle S^i. Further, set $\operatorname{rad} S^i := \sup\{|P - P^i|, P \in S^i\}$ and let $C_{fs} > 0$. In order to prove stability, we shall assume:

(3.1) For any sufficiently small $\epsilon > 0$, the operator A_ϵ is invertible and $\|A_\epsilon^{-1}\| \leq C_{fs}$ with C_{fs} a positive constant.

Note that A_ϵ is the finite-section operator defined by the left-hand side of (2.2). Thus assumption (3.1) means the stability of the finite-section method in the second step of the discretization of A. This finite-section method is analysed in [22], where it has been proved that (3.1) holds if and only if certain double-layer operators defined over the tangent cones are invertible. Sufficient for (3.1) is that each corner is rectangular and the graph of a Lipschitz function.[4] Another sufficient condition is that

$$\limsup_{r \to 0} \sup_{Q \in S} \frac{1}{2\pi} \int_{\{P \in S : |P - Q| \leq r\}} \frac{|n_P \cdot (Q - P)|}{|P - Q|^3} d_P S < 1. \tag{3.2}$$

This assumption was introduced also in [30] and can be checked using the fact that the integral of the double-layer kernel is a solid angle (cf. [19, §4.2.2]). In particular, (3.2) and therewith (3.1) hold for convex domains Ω. Now we have

THEOREM 3.1. *Suppose* (3.1) *holds. Then there is a* C *and an* ϵ_p $(1/2 > \epsilon_p > 0)$ *depending on* C_{fs} *and* C_1 *such that* $\|A_N^{-1}\| \leq C$ *holds whenever*

[4] A corner is called rectangular if there exist a neighbourhood U of the corner point and an orthogonal coordinate system such that the corner point has the coordinates (0,0,0) and $U \cap S$ is contained in the union of the three coordinate planes. The corner is the graph of a Lipschitz function if there exist a neighbourhood U of the corner point, an orthogonal coordinate system, and a Lipschitz function φ defined over the xy-plane such that $S \cap U$ coincides with $\{(x, y, \varphi(x, y)) : (x, y) \in \mathbf{R}^2\} \cap U$.

$$\frac{\operatorname{rad} S^i}{\operatorname{dist}(S^i, E^i)} \le \epsilon_p, \quad i = 1, \dots, N, \tag{3.3}$$

is satisfied.

Assumption (3.3) can be satisfied by choosing an appropriate partition of S_ϵ.[5] For the proof of Theorem 3.1 we shall need the following

LEMMA 3.2.

(i) *For any bounded function x over S, there holds*

$$\|W\chi x - W_N\chi x\| \le 5C_1\epsilon_p\|x\| + C_1 \max_{i=1,\dots,N} \sup_{P\in S^i} |x(P) - x(P^i)|. \tag{3.4}$$

(ii) *For any $x \in C(S)$,*

$$\max_{i=1,\dots,N} \sup_{P\in S^i} |W_N\chi x(P) - W_N\chi x(P^i)| \le C\sqrt{\epsilon_p}\|x\|. \tag{3.5}$$

(iii) *We have $\|W\chi I\| \le C_1$ and $\|W_N\chi I\| \le 5C_1$.*

Proof of Theorem 3.1. First we observe that $2[1 - W(\chi)]$ and $2[1 - W_N(\chi)]$ are bounded from below or, equivalently, that the operators of multiplication by the inverse functions are bounded operators. Namely, $A_\epsilon = 2[1 - W(\chi)]I + W\chi I$ is invertible by (3.1) and the operator $W\chi I$ is compact. Thus the multiplication operator $2[1 - W(\chi)]I$ is Fredholm with index zero and hence invertible; i.e., $2[1 - W(\chi)]$ is bounded from below. From Lemma 3.2 (i) with $x \equiv 1$, we get

$$\|W(\chi) - W_N(\chi)\| \le 5C_1\epsilon_p. \tag{3.6}$$

Thus, if ϵ_p is small enough, then

$$\left\| \frac{1}{2}[1 - W(\chi)]^{-1} 2[W(\chi) - W_N(\chi)] \right\| \le \frac{1}{2}$$

and $I + \frac{1}{2}[1 - W(\chi)]^{-1}2[W(\chi) - W_N(\chi)]I$ is invertible. Multiplying the last operator by $2[1 - W(\chi)]I$, we get that $2[1 - W_N(\chi)]I$ is invertible, too, and the function $2[1 - W_N(\chi)]$ is bounded from below. From these arguments we even conclude

$$\left\| \frac{1}{2}[1 - W(\chi)]^{-1} \right\| \le C, \quad \left\| \frac{1}{2}[1 - W_N(\chi)]^{-1} \right\| \le C, \tag{3.7}$$

where the constant depends only on C_{fs}. Hence, for the stability of A_N, it suffices to derive the invertibility of $\frac{1}{2}[1 - W_N(\chi)]^{-1}A_N = I + \frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi I$ from that of $\frac{1}{2}[1 - W(\chi)]^{-1}A_\epsilon = I + \frac{1}{2}[1 - W(\chi)]^{-1}2W\chi I$. This, however, follows as in [6]. Namely, let us introduce

$$T := \frac{1}{2}[1 - W(\chi)]^{-1}2W\chi I, \quad T_N := \frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi I$$

[5] The stability estimate and the convergence ratio of the two-grid iteration depend essentially on the ϵ_p of (3.3). To get a small ϵ_p the finite-section parameter ϵ should be large in comparison to the size of the subdomains S^i in the neighbourhood of strip *Str*. On the other hand, a larger ϵ leads to larger constants in (5.8).

and consider $\|(T - T_N)T_N\|$. We obtain

$$\begin{aligned}
&\left\|\left(\frac{1}{2}[1 - W(\chi)]^{-1}2W\chi I - \frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi I\right)\frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi I\right\| \\
&\leq \left\|\frac{1}{2}[1 - W_N(\chi)]^{-1}2[W(\chi) - W_N(\chi)]\frac{1}{2}[1 - W(\chi)]^{-1}2W\chi I\frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi I\right\| \\
&\qquad + \left\|\frac{1}{2}[1 - W_N(\chi)]^{-1}2(W\chi I - W_N\chi I)\frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi I\right\|. \qquad (3.8)
\end{aligned}$$

The first term on the right-hand side is smaller than $C\epsilon_p$ by (3.7), (3.6), and Lemma 3.2(iii). The second term can be estimated by Lemma 3.2(i) and (3.7). We get

$$\begin{aligned}
&\left\|\frac{1}{2}[1 - W_N(\chi)]^{-1}2(W\chi I - W_N\chi I)\frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi x\right\| \\
&\leq C \cdot 2 \cdot \left\{5C_1\epsilon_p\left\|\frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi x\right\|\right. \qquad (3.9) \\
&\left. + C_1 \max_{i=1,\ldots,N} \sup_{P\in S^i}\left|\left(\frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi x\right)(P) - \left(\frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi x\right)(P^i)\right|\right\},
\end{aligned}$$

where the term under the supremum can be estimated by

$$\begin{aligned}
&\left|\left(\frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi x\right)(P) - \left(\frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi x\right)(P^i)\right| \\
&\leq \left|\left\{\frac{1}{2}[1 - W_N(\chi)]^{-1}(P) - \frac{1}{2}[1 - W_N(\chi)]^{-1}(P^i)\right\}2(W_N\chi x)(P)\right| \\
&\qquad + \left|\frac{1}{2}[1 - W_N(\chi)]^{-1}(P^i)2\left\{(W_N\chi x)(P) - (W_N\chi x)(P^i)\right\}\right|. \qquad (3.10)
\end{aligned}$$

Together with (3.7), (3.6), and Lemma 3.2(iii) and (ii) we conclude

$$\begin{aligned}
&\left|\left(\frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi x\right)(P) - \left(\frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi x\right)(P^i)\right| \\
&\qquad \leq C\sqrt{\epsilon_p}\|x\|. \qquad (3.11)
\end{aligned}$$

Thus (3.9) and (3.11) lead to

$$\begin{aligned}
&\left\|\frac{1}{2}[1 - W_N(\chi)]^{-1}2(W\chi I - W_N\chi I)\frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi x\right\| \\
&\leq C\left\|2(W\chi I - W_N\chi I)\frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi x\right\| \leq C\sqrt{\epsilon_p}\|x\|. \qquad (3.12)
\end{aligned}$$

Equations (3.8) and (3.12) imply

$$\|(T - T_N)T_N\| \leq C\sqrt{\epsilon_p}. \qquad (3.13)$$

Taking into account that $(I+T)=\frac{1}{2}[1-W(\chi)]^{-1}A_\epsilon$ is invertible, we choose ϵ_p such that $\|(I+T)^{-1}\|\cdot\|(T-T_N)T_N\|\le\frac{1}{2}$ and obtain

$$\begin{aligned}\|[I+T+(T-T_N)T_N]^{-1}\| &\le \|(I+T)^{-1}\|\,\|\left[I+(I+T)^{-1}(T-T_N)T_N\right]^{-1}\| \\ &\le \|(I+T)^{-1}\|\cdot 2\le C. \end{aligned} \tag{3.14}$$

Furthermore, the operator

$$[I+T+(T-T_N)T_N]^{-1}[I+T-T_N] \tag{3.15}$$

is a left inverse of $(I+T_N)$. Since T_N is a finite-range operator, $(I+T_N)$ is Fredholm and its index is zero. Thus (3.15) is the inverse of $(I+T_N)$, and we get $\|(I+T_N)^{-1}\|\le C$. From $\frac{1}{2}[1+W_N(\chi)]^{-1}A_N=I+T_N$ and (3.7) we conclude $\|A_N^{-1}\|\le C$. □

Proof of Lemma 3.2(i). For $Q\in S$, we get

$$\begin{aligned} &W(\chi x)(Q)-W_N(\chi x)(Q) \\ &=\frac{1}{4\pi}\int_{S_\epsilon}\frac{n_P\cdot(P-Q)}{|P-Q|^3}x(P)d_PS-\frac{1}{4\pi}\sum_{i=1}^{N}\frac{n_{P^i}\cdot(P^i-Q)}{|P^i-Q|^3}x(P^i)\mu(S^i) \\ &=\frac{1}{4\pi}\sum_{i=1}^{N}\int_{S^i}\left\{\frac{n_P\cdot(P-Q)}{|P-Q|^3}x(P)-\frac{n_{P^i}\cdot(P^i-Q)}{|P^i-Q|^3}x(P^i)\right\}d_PS \\ &=\frac{1}{4\pi}\sum_{i=1}^{N}\int_{S^i}\frac{n_P\cdot(P-Q)}{|P-Q|^3}\{x(P)-x(P^i)\}d_PS \\ &\quad+\frac{1}{4\pi}\sum_{i=1}^{N}\int_{S^i}\left\{\frac{n_P\cdot(P-Q)}{|P-Q|^3}-\frac{n_{P^i}\cdot(P^i-Q)}{|P^i-Q|^3}\right\}x(P^i)d_PS. \end{aligned} \tag{3.16}$$

For $P,P^i\in S^i$, we conclude $n_P=n_{P^i}$ and $n_P\cdot(P-Q)=n_{P^i}\cdot(P^i-Q)$. Using this, we arrive at

$$\begin{aligned} |W(\chi x)(Q)-W_N(\chi x)(Q)| &\le C_1\max_{i=1,\dots,N}\sup_{P\in S^i}|x(P)-x(P^i)| \\ &\quad+\sum_{i=1}^{N}\|x\|\frac{1}{4\pi}\int_{S^i}\frac{|n_P\cdot(P-Q)|}{|P-Q|^3}\left|1-\frac{|P-Q|^3}{|P^i-Q|^3}\right|d_PS \\ &\le C_1\max_{i=1,\dots,N}\sup_{P\in S^i}|x(P)-x(P^i)| \\ &\quad+C_1\|x\|\max\sup_{P\in S^i}\left|1-\frac{|P-Q|^3}{|P^i-Q|^3}\right|, \end{aligned} \tag{3.17}$$

where the last maximum is taken over all $i=1,\dots,N$ such that S^i is not on the face of S that contains Q. Furthermore,

$$\left|1-\frac{|P-Q|^3}{|P^i-Q|^3}\right| \le \left\{1+\frac{|P-Q|}{|P^i-Q|}+\left(\frac{|P-Q|}{|P^i-Q|}\right)^2\right\}\left|1-\frac{|P-Q|}{|P^i-Q|}\right|$$

$$\le \left\{1+\left[|1-\frac{|P-Q|}{|P^i-Q|}|+1\right]+\left[|1-\frac{|P-Q|}{|P^i-Q|}|+1\right]^2\right\}\left|1-\frac{|P-Q|}{|P^i-Q|}\right|, \tag{3.18}$$

$$\left|1-\frac{|P-Q|}{|P^i-Q|}\right| \le \frac{|P^i-P|}{|P^i-Q|} \le \frac{\operatorname{rad} S^i}{\operatorname{dist}(S^i,E^i)} \le \epsilon_p < \frac{1}{2}. \quad \square \tag{3.19}$$

From (3.17)–(3.19) we conclude the validity of (i).

Proof of Lemma 3.2(ii). Let $\delta > 0$ and $S^{i,\delta} := \{P \in S_\epsilon \cap E^i$: for all $Q \in S^i$ there holds $|n_P \cdot (Q-P)|/|Q-P| \le \delta\}$. Furthermore, let Pl be an arbitrary plane not containing P^i and consider $Pl^{i,\delta} := \{P \in Pl : |n_P \cdot (P^i-P)|/|P^i-P| \le \delta\}$. Then the solid angle under which $Pl^{i,\delta}$ is seen from P^i is just $2\pi\delta$. Hence, (cf. [15] or [19, §4.2.2])

$$\frac{1}{4\pi}\int_{Pl^{i,\delta}} \frac{|n_P \cdot (P-P^i)|}{|P-P^i|^3} d_P S = \frac{1}{2}\delta. \tag{3.20}$$

Since $S^{i,\delta}$ is contained in less than M planes (M is the number of faces on S), we conclude

$$\frac{1}{4\pi}\int_{S^{i,\delta}} \frac{|n_P \cdot (P-Q)|}{|P-Q|^3} d_P S \le \frac{M}{2}\delta \tag{3.21}$$

for $Q \in S^i$. Now, for $P \in S^i$, we get

$$W_N\chi x(P) - W_N\chi x(P^i) = \frac{1}{4\pi}\sum_{k=1}^{N}\left\{\frac{n_{P^k}\cdot(P^k-P)}{|P^k-P|^3} - \frac{n_{P^k}\cdot(P^k-P^i)}{|P^k-P^i|^3}\right\} x(P^k)\mu(S^k)$$

$$= I_1 + I_2, \tag{3.22}$$

where I_1 is just the sum over the $k = 1, \dots, N$ with $S^k \subset S_\epsilon \setminus S^{i,\delta}$. The second term I_2 is the sum over $k = 1, \dots, N$ with $S^k \cap S^{i,\delta} \neq \emptyset$. If $R = R^k \in S^k \cap S^{i,\delta}$ and $Q \in S^k$, then

$$\frac{|n_Q \cdot (P^i-Q)|}{|P^i-Q|} = \frac{|n_R\cdot(P^i-R)|}{|P^i-R|}\cdot\frac{|P^i-R|}{|P^i-Q|} \le \delta\left(1+\frac{|R-Q|}{|P^i-Q|}\right)$$

$$\le \delta\left(1+\frac{\operatorname{rad} S^k}{\operatorname{dist}(S^k,E^k)}\right) \le \delta(1+\epsilon_p) \le 2\delta. \tag{3.23}$$

Consequently, $S^k \subseteq S^{i,2\delta}$ and we get

$$|I_2| \le \tag{3.24}$$

$$\|x\|\left\{\frac{1}{4\pi}\sum_{k:S^k\subseteq S^{i,2\delta}}\frac{|n_{P^k}\cdot(P^k-P)|}{|P^k-P|^3}\mu(S^k) + \frac{1}{4\pi}\sum_{k:S^k\subseteq S^{i,2\delta}}\frac{|n_{P^k}\cdot(P^k-P^i)|}{|P^k-P^i|^3}\mu(S^k)\right\}.$$

Analogously to the proof of (i) we conclude (cf. also (3.21))

$$\frac{1}{4\pi}\sum_{k:S^k\subseteq S^{i,2\delta}}\frac{|n_{P^k}\cdot(P^k-P)|}{|P^k-P|^3}\mu(S^k)=\frac{1}{4\pi}\int_{\{\cup_{S^k\subseteq S^{i,2\delta}}S^k\}}\frac{|n_R\cdot(R-P)|}{|R-P|^3}d_RS$$

$$+\frac{1}{4\pi}\sum_{k:S^k\subseteq S^{i,2\delta}}\int_{S^k}\left\{\frac{|n_{P^k}\cdot(P^k-P)|}{|P^k-P|^3}-\frac{|n_R\cdot(R-P)|}{|R-P|^3}\right\}d_RS$$

$$\leq\frac{1}{4\pi}\int_{S^{i,2\delta}}\frac{|n_R\cdot(R-P)|}{|R-P|^3}d_RS+C_15\epsilon_p\leq\frac{M}{2}2\delta+C\epsilon_p. \tag{3.25}$$

Hence,

$$|I_2|\leq 2\,\|x\|\{M\delta+C\epsilon_p\}. \tag{3.26}$$

On the other hand, we get

$$|I_1|\leq\|x\|\frac{1}{4\pi}\sum_{k:S^k\subseteq S_\epsilon\setminus S^{i,\delta}}\frac{|n_{P^k}\cdot(P^k-P^i)|}{|P^k-P^i|^3}\left|\left\{\frac{\frac{n_{P^k}\cdot(P^k-P)}{|P^k-P|^3}}{\frac{n_{P^k}\cdot(P^k-P^i)}{|P^k-P^i|^3}}-1\right\}\right|\mu(S^k), \tag{3.27}$$

$$\left\{\frac{\frac{n_{P^k}\cdot(P^k-P)}{|P^k-P|^3}}{\frac{n_{P^k}\cdot(P^k-P^i)}{|P^k-P^i|^3}}-1\right\}=\frac{\frac{n_{P^k}\cdot(P^k-P)}{|P^k-P|^3}-\frac{n_{P^k}\cdot(P^k-P^i)}{|P^k-P^i|^3}}{\frac{n_{P^k}\cdot(P^k-P^i)}{|P^k-P^i|^3}}$$

$$=\frac{\frac{n_{P^k}\cdot(P^k-P)}{|P^k-P|^3}-\frac{n_{P^k}\cdot(P^k-P^i)}{|P^k-P|^3}}{\frac{n_{P^k}\cdot(P^k-P^i)}{|P^k-P^i|^3}}$$

$$+\frac{\frac{n_{P^k}\cdot(P^k-P^i)}{|P^k-P|^3}-\frac{n_{P^k}\cdot(P^k-P^i)}{|P^k-P^i|^3}}{\frac{n_{P^k}\cdot(P^k-P^i)}{|P^k-P^i|^3}}$$

$$=\frac{n_{P^k}\cdot(P^i-P)}{n_{P^k}(P^k-P^i)}\cdot\frac{|P^k-P^i|^3}{|P^k-P|^3}+\left\{\frac{|P^k-P^i|^3}{|P^k-P|^3}-1\right\}. \tag{3.28}$$

If $\epsilon_p<\frac{\delta}{2}$ and S^k,S^i lie on different faces of S, then $S^k\subseteq S_\epsilon\setminus S^{i,\delta}$ implies that there exists a point $Q\in S^i$ such that

$$|n_{P^k}\cdot(Q-P^k)|\geq\delta|Q-P^k|,$$

$$|n_{P^k}\cdot(P^i-P^k)|\geq|n_{P^k}\cdot(Q-P^k)|-|P^i-Q|\geq\left\{\delta-\frac{|P^i-Q|}{|Q-P^k|}\right\}|Q-P^k|$$

$$\geq\left\{\delta-\frac{\operatorname{rad}S^i}{\operatorname{dist}(S^i,E^i)}\right\}|Q-P^k|\geq\{\delta-\epsilon_p\}|Q-P^k|\geq\frac{1}{2}\delta|Q-P^k|.$$

This and (3.18), (3.19) yield

$$\left|\left\{\frac{\frac{n_{P^k}(P^k-P)}{|P^k-P|^3}}{\frac{n_{P^k}(P^k-P^i)}{|P^k-P^i|^3}}-1\right\}\right| \le C\frac{|P^i-P|}{\delta|Q-P^k|}+C\epsilon_p$$

$$\le C\frac{\operatorname{rad} S^i}{\delta\operatorname{dist}(S^i,E^i)}+C\epsilon_p \le C\epsilon_p\left(\frac{1}{\delta}+1\right). \tag{3.29}$$

Now, analogously to (3.25), we conclude from (3.27), (3.29) that

$$|I_1| \le \|x\|C\left(\frac{1}{\delta}+1\right)\epsilon_p \le C\left\{\frac{1}{\delta}+1\right\}\epsilon_p\|x\|. \tag{3.30}$$

Together with (3.22) and (3.26) we get

$$|W_N\chi x(P)-W_N\chi x(P^i)| \le C\left\{\delta+\epsilon_p+\epsilon_p\left(\frac{1}{\delta}+1\right)\right\}\|x\|. \tag{3.31}$$

Let $\delta=\sqrt{\epsilon_p}$. Then we obtain

$$|W_N\chi x(P)-W_N\chi x(P^i)| \le C\sqrt{\epsilon_p}\|x\|. \tag{3.32}$$

□

Proof of Lemma 3.2(iii). The estimate $\|W\chi\| \le C_1$ follows from the definition of C_1. The second estimate is a consequence of the first and of (i). □

4. The convergence of the iteration process.

4.1. The two-grid iteration. Writing the steps (2.8)–(2.11) in one equation we arrive at

$$x^i = \{I-A_{N_c}^{-1}2W_N\chi I\}\frac{1}{2}[1-W_N(\chi)]^{-1}y+Op\ x^{i-1},$$

$$Op := A_{N_c}^{-1}2(W_N\chi I-W_{N_c}\chi I)\frac{1}{2}[1-W_N(\chi)]^{-1}2W_N\chi I.$$

Obviously, x_N is a fixed point of the iteration. Hence, from the last equation we get

$$(x^i-x_N) = Op\ (x^{i-1}-x_N). \tag{4.1}$$

However, the proof of Theorem 3.1 yields $\|A_{N_c}^{-1}\| \le C$ if

$$\frac{\operatorname{rad} S_c^i}{\operatorname{dist}(S_c^i,E_c^i)} \le \epsilon_p,\quad i=1,\dots,N_c. \tag{4.2}$$

Here E_c^i denotes the union of all faces of S not containing S_c^i. We get

$$\left\| 2(W_{N_c}\chi I - W_N\chi I)\frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi I \right\|$$

$$\leq \left\| 2(W\chi I - W_N\chi I)\frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi I \right\|$$
$$+ \left\| 2(W_{N_c}\chi I - W\chi I)\frac{1}{2}[1 - W_N(\chi)]^{-1}2W_N\chi I \right\|.$$

Similarly to (3.12) we conclude that both terms on the right-hand side of the last inequality are less than $C\sqrt{\epsilon_p}$ if (3.3) and (4.2) hold. Thus $\|Op\| \leq C\sqrt{\epsilon_p}$. If an arbitrary q with $0 < q < 1$ is given and ϵ_p is small enough, then $C\sqrt{\epsilon_p} \leq q < 1$; and from (4.1) we conclude

$$\|x^i - x_N\| \leq q\|x^{i-1} - x_N\|.$$

This leads to

THEOREM 4.1. *Suppose* (3.1) *holds, and choose an arbitrary* q *such that* $0 < q < 1$. *Then there is a small positive* ϵ_p *depending on* q *such that the two-grid iteration* (2.8)–(2.11) *converges for any triangulations* $S_\epsilon = \bigcup S^i, S_\epsilon = \bigcup S_c^i$ *satisfying* (3.3) *and* (4.2), *respectively. Moreover, there holds*

$$\|x^i - x_N\| \leq q^i\|x^0 - x_N\|. \tag{4.3}$$

Remark. Suppose no finite-section step is performed and, nevertheless, the operators A_N and A_{N_c} are stable. Instead of the truncation, assume that the partitions $S = \bigcup_{i=1}^N S^i$ and $S = \bigcup_{i=1}^{N_c} S_c^i$ coincide over the strip *Str* of width ϵ. Retain the definition of the set E^i from the beginning of §3 if i is the index of an S^i with $S^i \cap Str = \emptyset$. For $S^i \cap Str \neq \emptyset$, let E_0^i stand for the union of all faces on S that do not contain S^i and set $E^i := E_0^i \setminus Str$. Similarly, let us define E_c^i and consider (3.3) and (4.2) with this new definition of E^i and E_c^i. Then Theorem 4.1 remains true without the assumption (3.1). The proof is just the same since the restrictions of W_N and W_{N_c} to *Str* coincide, and thus these restrictions vanish in the difference $W_N - W_{N_c}$ and in the corresponding operator *Op*.

Remark. Obviously, Theorem 4.1 remains valid if we assume the stability of the coarse grid operator A_{N_c} instead of the assumption (3.1).

4.2. The Neumann iteration. For the iteration (2.12) we prove

THEOREM 4.2. *Suppose* (3.2) *holds and* $\kappa > 1$. *Then there is a* q, $0 < q < 1$, *and an* ϵ_p, $0 < \epsilon_p < \frac{1}{2}$, *such that, for any triangulation satisfying* (3.3), *the iteration* (2.12) *converges and there holds*

$$\|x^i - x_N\| \leq Cq^i\|x^0 - x_N\|. \tag{4.4}$$

Proof. Let us set $T_N := 2[\frac{1}{2} - W_N(\chi)]I + 2W_N\chi I$, $T := 2[\frac{1}{2} - W(\chi)]I + 2W\chi I$. Furthermore, let ρ_1 stand for the essential norm of the operator $2W$. It is well known that (3.2) implies $\rho_1 < 1$ (cf. [15], [30] or [19, §4.2.3]). Moreover, let ρ_2 denote the maximum of the absolute values of the eigenvalues of $2W$ that are different from

1. Then the well-known estimate $\rho_2 < 1$ follows, for example, from the proof of Theorem 12 in §1.3 of [19]. Hence, $\rho := \max\{\rho_1, \rho_2\} < 1$, and we can choose q with $\max\{\frac{2}{\kappa} - 1\ ,\ 1 + (\rho - 1)\frac{1}{\kappa}\} < q < 1$. Now it is not hard to verify that the iterative solutions of the Neumann iteration satisfy (4.1) with $Op := -\frac{1}{\kappa}\{T_N + (1-\kappa)I\}$. Thus all that we need to show is that the spectral radius of the operator Op is less than q. This, however, follows if we prove that the spectrum of T_N is contained in the set $Z := \{z \in C : |z| < \rho + \delta\} \cup \{z \in C : |z - 1| < \delta\}$ for any sufficiently small $\delta > 0$. Thus we have to show the invertibility of $A_{\lambda,N} := \{I - \lambda T_N\}$ for $\lambda^{-1} \in C \setminus Z$. To get this we observe that $A_\lambda := \{I - \lambda T\}$ is invertible for $\lambda^{-1} \in C \setminus Z$. Namely, the spectrum of $2W$ is contained in Z. The proofs of Theorems 2.1 and 3.2 of [22] yield the invertibility of the finite-section operator $A_{\lambda,\epsilon}$. Now consider the Nyström approximate operator $A_{\lambda,N}$ for A_λ and suppose (3.3) holds. Repeating the proof of Theorem 3.1, we conclude that, for sufficiently small ϵ_p, the operator $A_{\lambda,N}$ is invertible for any λ with $\lambda^{-1} \in C \setminus Z$. □

Remark. If the number ρ from the last proof is known explicitly, then the optimal choice for κ is $\kappa = (3 - \rho)/2$. Namely, in this case the lower bound $\max\{\frac{2}{\kappa} - 1\ ,\ 1 + (\rho - 1)\frac{1}{\kappa}\}$ of q is minimal. If ρ is unknown, then one should choose at least $1 < \kappa \leq \frac{3}{2}$.

Remark. As was mentioned in the last proof, $2W$ has a single eigenvalue $\lambda = 1$ and the rest of the spectrum is contained in $\{z \in \mathbf{C} : |z| \leq \rho\}$. Due to the singularity subtraction 1 is an eigenvalue of T_N, too. Moreover, let us choose any q with $\rho < q < 1$. Analogously to [24], in §5 one can prove that the corresponding eigenspace is of dimension one and the absolute values of the other eigenvalues are less than q provided the number ϵ_p in (3.3) is sufficiently small. We recommend the following iteration procedure: Start with $x^1 = \frac{y}{2}$ and set $x^i = y - (A_N - I)x^{i-1}$, $i = 2, 3, \ldots$. This iteration corresponds to the procedure (2.12), where $x^0 = 0$, $\kappa = 2$ for $i = 1$, and $\kappa = 1$ for $i = 2, \ldots$. The operator Op from (4.1) is equal to $\frac{1}{2}(I - T_N)$ for $i = 1$ and to $-T_N$ for $i = 2, \ldots$. Hence, the component of the error taken in the direction of the eigenfunction corresponding to eigenvalue 1 vanishes after the first step. Since this error component is zero and the other eigenvalues of T_N have absolute values less than q, the iteration converges and $\|x^i - x_N\| \leq Cq^i\|x^0 - x_N\|$. In the numerical example of §6.3 we have implemented this iteration.

Remark. Let us suppose again that ϵ_p is sufficiently small and that all the eigenvalues of T_N that are different from 1 are contained in $\{z \in \mathbf{C} : |z| < q\}$. In addition we suppose that T_N is diagonalizable. Then (cf. [25]) the GMRES is convergent and $\|x^i - x_N\| \leq Cq^i\|x^0 - x_N\|$.

5. Rates of convergence and complexity.

5.1. The error estimate and the complexity for Gaussian elimination. In order to get rates of convergence we need more information about the triangulation. Following [21], we could introduce special triangulations for an arbitrary polyhedron Ω. For the sake of simplicity, however, let us define the partitions only for the case of the cube $\Omega = (0,1) \times (0,1) \times (0,1)$.[6] Since the faces of the cube are squares, we can use rectangles in the partition of S_ϵ. Moreover, by symmetry it is enough to define the partition over one face, e.g., over $F := [0,1] \times [0,1] \simeq [0,1] \times [0,1] \times \{0\}$. We choose the fixed-grid parameters $\alpha \geq 1, i_0 \in \mathbf{Z}_+$ and define, for any integer $n > i_0$, the set $S_\epsilon = S_n$ and the partition $S_n = \bigcup_{i=1}^{N_n} S^i$ depending on n. To get these partitions over

[6] All the results of this section remain true if the boundary S is locally the graph of a Lipschitz function, if the Nyström method is stable, and if the special grids introduced in §4 of [21] are used.

F, we first divide F into n "strips" $F_i, i = 0, \dots, n-1$, parallel to the sides of F with distances $\frac{1}{2}(\frac{i}{n})^\alpha$ to the boundary (cf. Fig. 5.1):

$$F_i := \Big\{(x,y) \in F : \frac{1}{2}\left(1 - \left(\frac{i+1}{n}\right)^\alpha\right) < \max\left\{\left|\frac{1}{2} - x\right|, \left|\frac{1}{2} - y\right|\right\} < \frac{1}{2}\left(1 - \left(\frac{i}{n}\right)^\alpha\right)\Big\}. \tag{5.1}$$

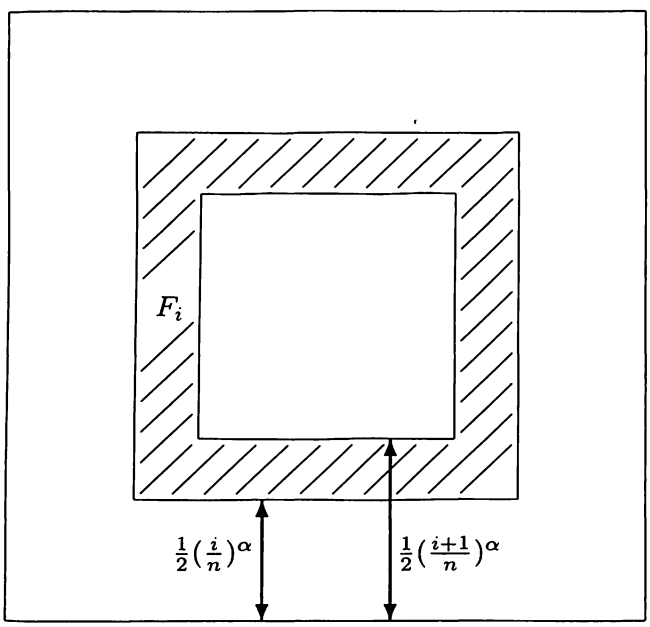

FIG. 5.1. *The strip F_i.*

The number α is the grading parameter. Choosing a high α results in very small strips near the sides of F, i.e., near the set of edge points. We introduce $\epsilon := (i_0/n)^\alpha$ and obtain $F \cap Str = \bigcup_{i=0}^{i_0-1} F_i$ and $S_n = S \setminus Str$. Thus the parameter i_0 defines the size of the strip that we cut off in the finite-section step. Now we only have to divide the rest of the strips $F_i, i = i_0, \dots, n-1$, into rectangles. In order to satisfy (3.3) for a small ϵ_p and to obtain a small number of subdivision domains, we divide F_i into rectangles, where the lengths of the sides are nearly equal to the width of the strip. For symmetry reasons, it is enough to give this partition of F_i over the part

$$F_i' := \left\{(x,y) \in \mathbf{R}^2 : \frac{1}{2}\left(\frac{i}{n}\right)^\alpha < x < \frac{1}{2}\left(\frac{i+1}{n}\right)^\alpha, \frac{1}{2}\left(\frac{i}{n}\right)^\alpha < y < 1 - \frac{1}{2}\left(\frac{i}{n}\right)^\alpha\right\}.$$

We set $F_i' = \bigcup_{j=1}^{k} F_{i,j}$ with

$$F_{i,j} := \left\{(x,y) \in \mathbf{R}^2 : \frac{1}{2}\left(\frac{i}{n}\right)^\alpha < x < \frac{1}{2}\left(\frac{i+1}{n}\right)^\alpha, t_j < y < t_{j+1}\right\}, \tag{5.2}$$

$$t_1 = \frac{1}{2}\left(\frac{i}{n}\right)^\alpha, t_2 = \frac{1}{2}\left(\frac{i+1}{n}\right)^\alpha, t_k = 1 - \frac{1}{2}\left(\frac{i+1}{n}\right)^\alpha, t_{k+1} = 1 - \frac{1}{2}\left(\frac{i}{n}\right)^\alpha, \tag{5.3}$$

$$t_j := t_2 + (j-2)\frac{t_k - t_2}{k-2},\ j = 3, 4, \dots, k-1.$$

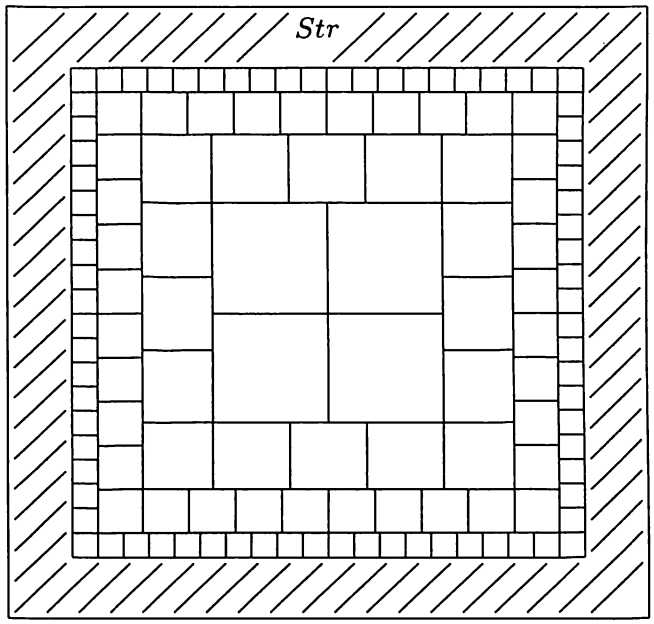

FIG. 5.2. *Partition of F.*

In order to get that the $F_{i,j}$ are almost squares, we choose k such that

$$t_j - t_{j-1} = \frac{t_k - t_2}{k-2} = \frac{1-(\frac{i+1}{n})^\alpha}{k-2} \approx \frac{1}{2}\left(\left(\frac{i+1}{n}\right)^\alpha - \left(\frac{i}{n}\right)^\alpha\right), \tag{5.4}$$

$$k := 2 + \left\lfloor \frac{1-(\frac{i+1}{n})^\alpha}{\frac{1}{2}((\frac{i+1}{n})^\alpha - (\frac{i}{n})^\alpha)} \right\rfloor.$$

Now the partition $S_n = \bigcup_{i=1}^{N_n} S^i$ is the union over all rectangles $F_{i,j}$ for all strips F_i and all faces of S (cf. Fig. 5.2).

Summing up the numbers k from (5.4) for $i = i_0, \ldots, n-1$, we obtain that the number N_n is of order

$$N_n \sim \begin{cases} n^\alpha & \text{if} \quad \alpha > 2, \\ n^2 \log n & \text{if} \quad \alpha = 2, \\ n^2 & \text{if} \quad 2 > \alpha. \end{cases} \tag{5.5}$$

Condition (3.3) is fulfilled for sufficiently large i_0 since

$$\frac{\operatorname{rad} S^i}{\operatorname{dist}(S^i, E^i)} \le C \frac{\frac{1}{2}((\frac{i_0+1}{n})^\alpha - (\frac{i_0}{n})^\alpha)}{\frac{1}{2}(\frac{i_0}{n})^\alpha} \le C \frac{\alpha(i_0+1)^{\alpha-1}}{i_0^\alpha} \le C \frac{1}{i_0}. \tag{5.6}$$

In other words, if i_0 is large enough, then Theorem 3.1 implies the stability of the Nyström method (2.3) over the partition $S_n = \bigcup_{i=1}^{N_n} S^i$. On the other hand, a larger i_0 will lead to a larger error in the quadrature. Thus we choose i_0 to be the smallest number such that (2.3) is stable. Now suppose the right-hand side y of our integral equation is continuous on S and C^∞ on each face of S. Then (cf. [19, §5.1.4]) the solution x fulfills

$$\sup_{P\in S} |x(P)| \le C, \quad |\nabla^l x(P)| \le C\mathrm{dist}(P,e)^{\delta-l} \quad (l=1,\ldots), \tag{5.7}$$

where e is the edge nearest to P and $0 < \delta < 1$ is a certain number. This number δ depends on the geometry of S. Its determination for arbitrary S is a hard problem since it requires the computation of eigenvalues for certain boundary value problems over spherical domains.

THEOREM 5.1. *Let $\alpha \ge 1$. Suppose* (3.1) *holds, and choose i_0 such that* (3.3) *is satisfied with $0 < \epsilon_p$, where ϵ_p is taken from Theorem* 3.1. *If the function y is continuous on S and C^∞ on each face of S and if x_N is the solution of* (2.3), *then*

$$\sup_{i=1,\ldots,N_n} |x_N(P^i) - x(P^i)| \le C \begin{cases} n^{-2} & \text{if} \quad \alpha > \frac{2}{\delta}, \\ n^{-2}\log n & \text{if} \quad \alpha = \frac{2}{\delta}, \\ n^{-\alpha\delta} & \text{if} \quad \alpha < \frac{2}{\delta}. \end{cases} \tag{5.8}$$

Here δ is the exponent appearing in (5.7) *and C denotes a constant depending only on S, i_0, and α.*

Proof. Let us define the piecewise interpolation projection P_N by

$$P_N f(P) := \begin{cases} f(P^i) & \text{if } P \in S^i \text{ for an } i = 1,\ldots,N_n, \\ 0 & \text{if } P \in Str. \end{cases}$$

Since the $W_N\chi f$ depends on $\{f(P^i)\}$ only, we get $W_N\chi I = W_N\chi P_N$ and, for the multiplication operator $2[1 - W_N(\chi)]I$, we arrive at $P_N 2[1 - W_N(\chi)]P_N = P_N 2[1 - W_N(\chi)]I$. Thus

$$\begin{aligned} A_N = 2[1 - W_N(\chi)]I + 2W_N\chi I &= P_N \{A_N|_{\mathrm{im}\, P_N}\} P_N \\ &+ (I - P_N)\{2[1 - W_N(\chi)]I\}(I - P_N) + (I - P_N)\{A_N|_{\mathrm{im}\, P_N}\} P_N. \end{aligned}$$

From this equation we conclude that A_N is invertible if and only if $\{P_N A_N|_{\mathrm{im}\, P_N}\}$ is invertible, and

$$\| \{P_N A_N|_{\mathrm{im}\, P_N}\}^{-1} \| \le C \| A_N^{-1} \|.$$

Using the boundedness of $\| \{P_N A_N|_{\mathrm{im}\, P_N}\}^{-1} \|$, the proof of Theorem 5.1 follows analogously to that of Theorem 4.1 in [21]. The only difference is that instead of (4.5) in [21] one has to use

$$|x(P) - x(P^j)| \le |P - P^j|^\delta,$$

which follows from the formulae (5.7). Note that only the last inequality is used in the considerations following (4.5) of [21]. □

Now let us suppose we solve (2.4) using Gaussian elimination. Then the number of operations is of order N_n^3. In order to get an error smaller than a prescribed positive ϵ_C, we have to choose (cf. (5.8)) $n \sim \epsilon_C^{-1/2}$ if $\alpha > 2/\delta$, $n \sim \epsilon_C^{-1/2}\log^{1/2}(\epsilon_C^{-1})$ if $\alpha = \frac{2}{\delta}$, and $n \sim \epsilon_C^{-1/(\alpha\delta)}$ if $\alpha < \frac{2}{\delta}$. Consequently, the number of complexity *comp*, i.e., the number of operations to achieve an error less than ϵ_C, is given by (cf. (5.5))

$$comp \sim \begin{cases} \epsilon_C^{-6/(\alpha\delta)} & \text{if} \quad 1 \le \alpha < 2, \\ \epsilon_C^{-3/\delta} \log^3(\epsilon_C^{-1}) & \text{if} \quad \alpha = 2, \\ \epsilon_C^{-3/\delta} & \text{if} \quad 2 < \alpha < \frac{2}{\delta}, \\ \epsilon_C^{-3/\delta} \log^{3/\delta}(\epsilon_C^{-1}) & \text{if} \quad \alpha = \frac{2}{\delta}, \\ \epsilon_C^{-3\alpha/2} & \text{if} \quad \frac{2}{\delta} < \alpha. \end{cases} \tag{5.9}$$

Note that these estimates of complexity and those of the next subsections are of an asymptotic nature. The constants hidden in the notation $\sim$ depend on S and on the parameter α. In other words, the optimal choice of α will depend on n and S. However, for the sake of simplicity, we suppose that n is very large such that the constants are of no importance.

5.2. The complexity of the two-grid method. Now let us consider the two-grid iteration (2.8)–(2.11). We choose the coarse grid as follows: First we choose a fixed parameter $\eta > 1$. As above we give the grid only over one face of S, i.e., on the square F. We introduce the strips $F_{c,i}$ with distance $\eta_i/2$ to the boundary, where $\eta_i := (i_0/n)^\alpha \eta^i$, $i = 0, \dots, i_1$. In other words we set

$$F_{c,i} := \left\{ (x,y) \in F : \frac{1}{2}(1-\eta_{i+1}) < \max\left\{ \left|\frac{1}{2} - x\right|, \left|\frac{1}{2} - y\right| \right\} < \frac{1}{2}(1-\eta_i) \right\},$$

$$i = 0, 1, \dots, i_1, \; i_1 := \left[\alpha \frac{\log n/i_0}{\log \eta} \right], \; \eta_{i_1+1} := 1 \tag{5.10}$$

and get a similar partition into strips as for the fine grid, only with a geometric grading. Since η is a fixed number, the maximal width of the strips does not tend to zero if n tends to infinity. Analogously to the fine grid, we define the further partition only over

$$F'_{c,i} := \left\{ (x,y) \in \mathbf{R}^2 : \frac{1}{2}\eta_i < x < \frac{1}{2}\eta_{i+1}, \; \frac{1}{2}\eta_i < y < 1 - \frac{1}{2}\eta_i \right\}.$$

We set

$$F_{c,i,j} := \left\{ (x,y) \in \mathbf{R}^2 : \frac{1}{2}\eta_i < x < \frac{1}{2}\eta_{i+1}, \; t_{c,j} < y < t_{c,j+1} \right\},$$

$$t_{c,1} := \frac{1}{2}\eta_i, \; t_{c,2} := \frac{1}{2}\eta_{i+1}, \; t_{c,k} := 1 - \frac{1}{2}\eta_{i+1}, \; t_{c,k+1} := 1 - \frac{1}{2}\eta_i, \tag{5.11}$$

$$t_{c,j} := t_{c,2} + (j-2)\frac{t_{c,k} - t_{c,2}}{k-2}, \; j = 3, \dots, k-1, \tag{5.12}$$

where k is chosen such that $F_{c,i,j}$ is nearly a square, i.e.,

$$t_{c,j} - t_{c,j-1} = \frac{t_{c,k} - t_{c,2}}{k-2} = \frac{1-\eta_{i+1}}{k-2} \approx \left[\frac{1}{2}\eta_{i+1} - \frac{1}{2}\eta_i \right], \tag{5.13}$$

$$k := 2 + \left[\frac{1-\eta_{i+1}}{\frac{1}{2}\eta_{i+1} - \frac{1}{2}\eta_i} \right]. \tag{5.14}$$

Now the partition $S_n = \bigcup_{i=1}^{N_{c,n}} S_c^i$ is the union over all rectangles $F_{c,i,j}$ for all strips $F_{c,i}$ and all faces of S.

In every strip $F_{c,i}$ we have approximately $O(\{\frac{1}{2}(i_0/n)^\alpha \eta^{i+1} - \frac{1}{2}(i_0/n)^\alpha \eta^i\}^{-1})$ subdomains. Thus the number $N_{c,n}$ of all subdivision domains is of order n^α. For η sufficiently close to one, condition (4.2) is fulfilled since

$$\frac{\operatorname{rad} S_c^i}{\operatorname{dist}(S_c^i, E_c^i)} \leq C \frac{\frac{1}{2}(\frac{i_0}{n})^\alpha \eta^{i+1} - \frac{1}{2}(\frac{i_0}{n})^\alpha \eta^i}{\frac{1}{2}(\frac{i_0}{n})^\alpha \eta^i} \leq C(\eta - 1). \tag{5.15}$$

The LU-factorization of the coarse-grid matrix corresponding to A_{N_c} requires $O(N_{c,n}^3)$ operations, and each iteration step of the two-grid algorithm requires $O(N_n^2)$. To obtain an error less than ϵ_C we need about $\log \epsilon_C^{-1}$ iterations. For the complexity of this iteration, we conclude from (5.5), (5.8), and $N_{c,n} \sim n^\alpha$ that

$$comp \sim \begin{cases} \epsilon_C^{-4/(\alpha\delta)} \log(\epsilon_C^{-1}) & \text{if} \quad 1 \leq \alpha \leq \frac{4}{3}, \\ \epsilon_C^{-3/\delta} & \text{if} \quad \frac{4}{3} < \alpha < \frac{2}{\delta}, \\ \epsilon_C^{-3/\delta} \log^{3/\delta}(\epsilon_C^{-1}) & \text{if} \quad \alpha = \frac{2}{\delta}, \\ \epsilon_C^{-3\alpha/2} & \text{if} \quad \frac{2}{\delta} < \alpha. \end{cases} \tag{5.16}$$

Note that the factor $\log(\epsilon_C^{-1})$ in the case $1 \leq \alpha \leq \frac{4}{3}$ can be removed if we solve the Nyström equation over a sequence of meshes and start the iteration with the solution from a coarser grid as an initial function. In the case of a one-dimensional boundary this variant is analysed by Atkinson and Graham [5]. We also note that in practical computations the coarse-grid equation can be solved by another iteration method. This would lead to better orders of complexity.

5.3. The complexity of the Neumann iteration. If we use the procedure proposed by Wendland, then (cf. Theorem 4.2) we need about $C \log(\epsilon_C^{-1})$ iterations in order to get an error less than ϵ_C, where C depends on the geometry of S, only. Hence, we need $C N_n^2 \log(\epsilon_C^{-1})$ operations and the complexity is given by (cf. (5.8), (5.5))

$$comp \sim \begin{cases} \epsilon_C^{-4/(\alpha\delta)} \log(\epsilon_C^{-1}) & \text{if} \quad 1 \leq \alpha < 2, \\ \epsilon_C^{-2/\delta} \log^3(\epsilon_C^{-1}) & \text{if} \quad \alpha = 2, \\ \epsilon_C^{-2/\delta} \log(\epsilon_C^{-1}) & \text{if} \quad 2 < \alpha < \frac{2}{\delta}, \\ \epsilon_C^{-2/\delta} \log^{1+2/\delta}(\epsilon_C^{-1}) & \text{if} \quad \alpha = \frac{2}{\delta}, \\ \epsilon_C^{-\alpha} \log(\epsilon_C^{-1}) & \text{if} \quad \frac{2}{\delta} < \alpha. \end{cases} \tag{5.17}$$

Again, one factor $\log(\epsilon_C^{-1})$ can be dropped if a sequence of partition is used (cf. the end of §5.2).

Summarizing the complexity results, we can say the following: If one uses Gaussian elimination for the solution of the system of linear equations, then the best choice of α is to take α a little bit larger than two or, roughly speaking, to set $\alpha = 2$. With this choice the complexity is reduced from $C\epsilon_C^{-6/\delta}$ for the uniform mesh with $\alpha = 1$ to $C\epsilon_C^{-3/\delta}$ for the graded mesh defined by $\alpha \in (2, \frac{2}{\delta})$. Especially, the optimal choice of α is independent of the parameter δ appearing in (5.7). The Neumann iteration with the same α results even in a complexity of about $C\epsilon_C^{-2/\delta}$. However, if the geometry

of S is complicated, then C is large and the Neumann iteration is slow. In this case we recommend the two-grid iteration, where the convergence speed of the iteration can be improved by using appropriate meshes. This method together with the choice $\alpha = \frac{4}{3}$ gives the same asymptotic rate as obtained by Nyström's method together with Gaussian elimination and $\alpha = 2$. The constant in the asymptotic estimate for the two-grid algorithm, however, seems to be much smaller. If we compare two-grid iteration and Gaussian elimination for the same α and for large n, then the complexity of the two-grid iteration is smaller since the LU-factorization is performed for a smaller system of equations.

6. Numerical examples.

6.1. The error of the Nyström method. In the numerical examples of §6 we shall consider the double-layer potential equation over the boundary of three different domains Ω_i, $i = 1, 2, 3$. The first one is the convex cube $\Omega_1 := [0,1] \times [0,1] \times [0,1]$. Besides this we consider the L-block Ω_2, i.e., the direct product of the L-shaped domain $([0,1] \times [0,0.5]) \cup ([0.25,1] \times [0.5,1])$ in the xz-plane multiplied by the interval $[0,1]$ in y-direction. Note that Ω_2 is rectangular and locally the graph of a Lipschitz function. Our third domain Ω_3 is the polyhedron of Fig. 6.1, where we have chosen $P_1 := (0, 0, \frac{1}{\sqrt{2}})$, $P_2 := (0,0,0)$, $P_3 := (-1,-1,-\lambda)$, $P_4 := (1,-1,\lambda)$, $P_5 := (1,1,-\lambda)$, and $P_6 := (-1,1,\lambda)$ with $\lambda := 1$.[7] This polyhedron Ω_3 does not fulfil any sufficient condition for the stability of our Nyström method.

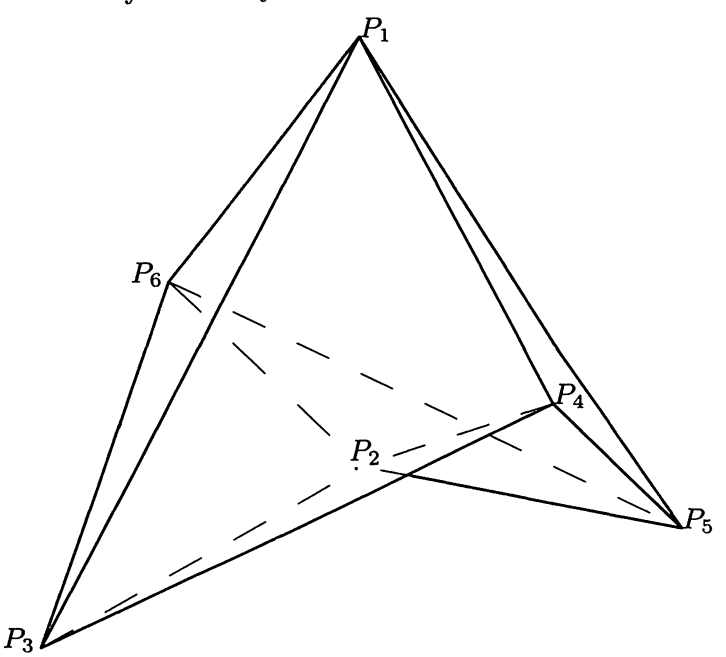

FIG. 6.1. *Polyhedron* Ω_3.

In our first example let us consider method (2.3) over Ω_2 for the continuous but nonharmonic data $y(P) = 6|P - (1,0,0)|^{-2}$. The "supremum norm" error $SER = SER_n$ depending on the values n, i_0, and α is given in Table 6.1. For simplicity, the supremum norm is just the maximum of the error taken over the six points (0.25,0,0.5), (0.25,0.5,0.5), (0.625,0,0.25), (0.25,0.1,0.5), (0.325,0,0.45), and

[7] In order to get a nice and clear picture, we have drawn Ω_3 in Fig. 6.1 with a smaller λ.

TABLE 6.1
Method (2.4), Ω_2, *supremum norm error, error of Dirichlet solution at* (0.5,0.5,0.5).

α	i_0	n	N_n	SER_n	EX_n	ERR	α	i_0	n	N_n	SER_n	EX_n	ERR
1	0	3	406	0.118	0.24	0.0044	$\frac{4}{3}$	1	3	172	0.148	0.74	0.00048
		4	736	0.0559	2.59	0.00225			4	388	0.116	0.85	0.00029
		5	1164	0.072	-1.14	0.00128			5	730	0.095	0.87	0.00059
		6	1688	0.0177	7.69	0.000780			6	1190	0.081	0.93	0.00072
		7	2316	0.0416	-5.53	0.000475			7	1710	0.069	0.99	0.00076
		8	3034	0.00078	29.77	0.000283			8	2406	0.060	1.00	0.00077
1	1	3	127	0.170	0.64	0.00061	2	0	3	646	0.075	-1.82	0.0045
		4	406	0.144	0.59	0.0014			4	1310	0.0384	2.33	0.0024
		5	736	0.125	0.63	0.0016			5	2234	0.0205	2.82	0.0014
		6	1164	0.111	0.65	0.0016							
		7	1688	0.100	0.68	0.0015							
		8	2316	0.091	0.70	0.0014							
$\frac{4}{3}$	0	3	458	0.1151	-0.31	0.0044	2	1	3	172	0.109	1.13	0.0020
		4	836	0.0722	1.62	0.00245			4	426	0.0741	1.35	0.00086
		5	1370	0.0445	2.17	0.00136			5	824	0.0517	1.61	0.00043
		6	2030	0.0260	2.94	0.00083			6	1384	0.0519	-0.02	0.00015
		7	2746	0.0136	4.18	0.00051			7	2102	0.0483	0.46	0.00002
									8	2982	0.0397	1.47	0.00009
									9	4082	0.0341	1.29	0.00015
							2	2	4	172	0.180	1.10	0.0031
									5	388	0.142	1.06	0.0022
									6	744	0.114	1.17	0.0016
									7	1218	0.0938	1.29	0.0013
									8	1808	0.0779	1.39	0.0011
									9	2576	0.0656	1.46	0.00099

TABLE 6.2
Method (2.4), Ω_1, *error of Dirichlet solution at the midpoint.*

α	i_0	n	N_n	N_u	ERR	α	i_0	n	N_n	N_u	ERR
1	0	2	96	96	0.014	2.5	1	2	24	96	0.00019
		3	216	216	0.0063			3	144	216	0.00099
		4	384	384	0.0036			4	408	384	0.00038
		5	600	600	0.0023						
4/3	0	2	120	96	0.014	3.5	1	2	24	96	0.0039
		3	264	216	0.0059			3	192	216	0.0017
		4	504	384	0.0034						
4/3	1	2	24	96	0.0028						
		3	96	216	0.0082	3.5	2	3	24	216	0.0099
		4	240	384	0.0081			4	144	384	0.0055
		5	408	600	0.0072						

(0.025,0,0.25). Moreover, to determine this error we have replaced the true solution by an approximate solution computed with large n ($n = 30$). By $EX = EX_n$ we denote the following estimate for the order of convergence:

$$EX := -\frac{\log SER_n - \log SER_{n-1}}{\log n - \log(n-1)}.$$

Thus $SER_n \sim n^{-EX}$. The mesh for the Nyström method in Table 6.1 is defined analogously to §5.1. Note that the partition $S_\epsilon = \bigcup_{i=1}^{N_n} S^i$ of §5.1 is determined by the parameters α, i_0, and n, where α is the degree of mesh refinement near the edges. The integer n denotes the total number of strips and i_0 the number of strips that are neglected in the quadrature rules. High values of α lead to small subdivision domains

near the edges and to greater subdivision domains around the midpoints of the faces of $S = \partial\Omega_2$. In order to keep the last subdomains small, we have used an idea of Kress (cf. (2.2) in [18]) and have replaced[8] the function $\frac{i}{n} \mapsto (\frac{i}{n})^\alpha$ in the definition of the strips F_i by a function $\frac{i}{n} \mapsto \varphi(\frac{i}{n})$, where $\varphi : [0,1] \to [0,1]$ behaves like $t \mapsto t^\alpha$ for t near to 0 and $C_\varphi := \max_{0\le t\le 1} |\varphi'(t)|$ is as small as possible. Note that the maximal side length of the subdomains in the strip F_i is equal to $\frac{1}{2}[\varphi(\frac{i+1}{n}) - \varphi(\frac{i}{n})]$ and can be estimated by $\frac{1}{2n} C_\varphi$. Especially, we have set

$$\varphi(t) := \frac{\{t[1+\lambda_1 - \lambda_1 t]\}^\alpha}{\{t[1+\lambda_1-\lambda_1 t]\}^\alpha + \{(1-t)[1+\lambda_1-\lambda_1(1-t)]\}}.$$

The parameter λ_1 is 0 for $\alpha = 1$, 0.105 for $\alpha = \frac{4}{3}$, 0.28 for $\alpha = 2$, and 0.48 for $\alpha = 3$. Now if U is the solution of the Dirichlet problem in Ω_2 with the boundary value $U(P) = 3|P - (1,0,0)|^{-2}$ $(P \in S)$, then U admits the representation

$$U(Q) = \frac{1}{4\pi} \int_S \frac{n_P \cdot (Q-P)}{|Q-P|^3} x(P) d_P S, \tag{6.1}$$

where x is the solution of the double-layer potential equation $Ax = y$. Substituting x_N into this representation formula and computing the integral via quadrature rule, we arrive at the formula

$$U_N(Q) := \frac{1}{4\pi} \sum_{i=1}^{N_u} \frac{n_{P_u^i} \cdot (Q - P_u^i)}{|Q - P_u^i|^3} x_N(P_u^i) \mu(S^i). \tag{6.2}$$

Here $S = \bigcup_{i=1}^{N_u} S_u^i$ denotes the uniform partition of §5.1 defined with $\alpha = 1$ and $i_0 = 0$. The error $ERR := |U(Q) - U_N(Q)|$ for $Q = (0.5, 0.5, 0.5)$ is given in the last column of Table 6.1. If we compare the supremum norm errors in Table 6.1 for $i_0 = 0$ and equal numbers N_n, then the choice $\alpha = 2$, $n = 5$ is better than $\alpha = \frac{4}{3}$, $n = 6$ or $\alpha = 1$, $n = 7$. The best approximate values for $U(0.5, 0.5, 0.5)$ correspond to the parameters $\alpha = 2$, $i_0 = 1$. Thus graded meshes lead to better approximations. However, since the convergence orders EX oscillate, the effect of mesh grading is not as clear as expected in view of Theorem 5.1. For small n, $\alpha = 1$ is still a good choice. Moreover, we have no explanation for the high orders of convergence obtained in the case $i_0 = 0$. The first term in the asymptotics of the solution seems to be due to the edge singularity, i.e., the exponent δ appearing in (5.7) is equal to $\frac{2}{3}$. Thus EX should tend to $\frac{2}{3} \cdot \alpha$. The values EX, however, are close to $\frac{2}{3} \cdot \alpha$ only for $i_0 > 0$. The smallest supremum norm errors are obtained in the case $i_0 = 0$ and we recommend choosing $i_0 = 0$ whenever the method (2.3) is stable with this choice. For the pointwise behaviour of the error, we refer to the paper [21].

Next let us consider the cube Ω_1, $S := \partial\Omega_1$, and the harmonic function

$$U(Q) := \frac{1}{2}\left\{ \frac{1}{|Q - (1.5, 0.5, 0.5)|} + \frac{1}{|Q - (-0.5, 0.5, 0.5)|} \right\}, \quad Q \in \Omega_1. \tag{6.3}$$

[8] If n is very small and the strip *Str* is neglected, then the mesh after the replacement is nearly uniform. Therefore, we have used the original function $\frac{i}{n} \mapsto (\frac{i}{n})^\alpha$ for the computations in Tables 6.2 and 6.7.

Then $U(0.5, 0.5, 0.5) = 1$ and U is given by (6.1), where x is the solution of $Ax = y := 2U|_S$. Table 6.2 contains some results for the error $ERR := |U_N(0.5, 0.5, 0.5) - U(0.5, 0.5, 0.5)|$ depending on α, i_0, and n. The choice $i_0 = 1$ yields smaller systems of equations (cf. N_n in Table 6.2) and smaller errors ERR in comparison to the choice $i_0 = 0$. We have observed this surprising result also for Ω_2 but not for all boundary values $U|_S$.

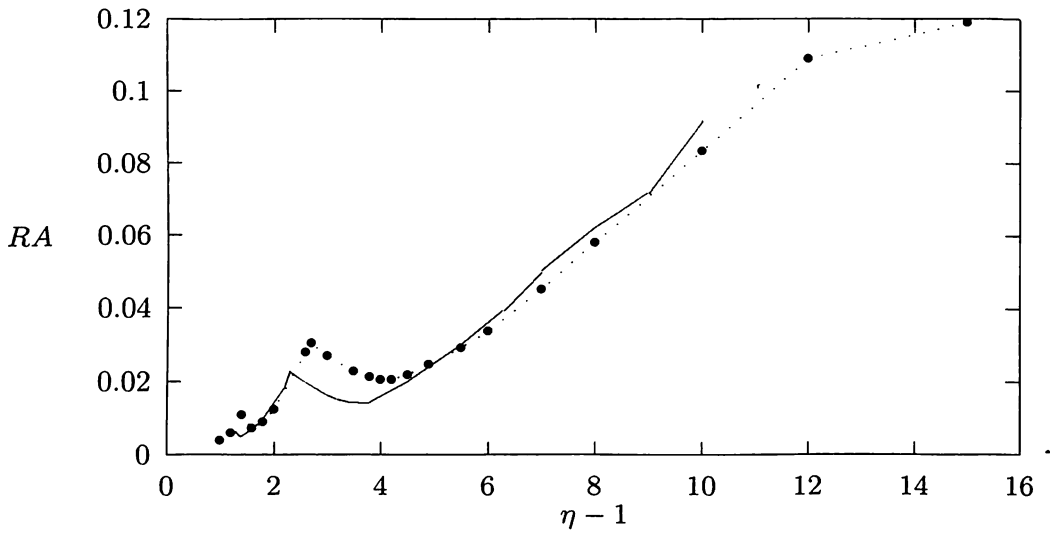

FIG. 6.2. *Convergence factor RA of the two-grid method, Ω_2, continuous line: $n = 6$, dotted line: $n = 7$, neglect of Str, $i_0 = 1$, $\alpha = \frac{4}{3}$.*

6.2. Convergence of the two-grid iteration. Let us consider Ω_2 and the two-grid method (2.8)–(2.11), where the coarse grid is defined as in §5.2. The choice $i_0 = 1$ will be sufficient to get a fast iteration procedure. By (5.15) we get $\epsilon_p = C(\eta - 1)$, and from §4.1 we conclude $\|Op\| < C\sqrt{\epsilon_p} = C\sqrt{\eta - 1}$. To confirm this dependence, let us introduce the estimate RA for $\|Op\|$ by setting

$$RA := \sup_{i=2,\dots} \frac{\|x^i - x_N\|}{\|x^{i-1} - x_N\|}, \tag{6.4}$$

$$\|x^i - x_N\| := \sup_{j=1,\dots,N_n} |x^i(P^j) - x_N(P^j)|. \tag{6.5}$$

Some values for RA are given in Fig. 6.2. Indeed they show that RA is smaller for smaller $(\eta - 1)$. We even get a satisfactory RA if the coarse grid is obtained by dividing each face $F \setminus Str$ (cf. §5.2.) into four equal squares. Since this choice leads to a small number N_c and a small number of necessary operations in the iteration process, we have used this coarse grid for the computations in Table 6.7. If we do not neglect the strip Str, then the convergence factor RA is much greater (equal to 0.4). On the other hand, if we do not neglect Str but choose the coarse grid equal to the fine one over Str, then RA is small again. Some of the values of RA are presented in Fig. 6.3. Again, we get a satisfactory RA if the coarse grid is chosen equal to the fine

one over Str and equal to the four uniform squares over each face $F \setminus Str$. This choice has been used in the computations for Tables 6.3 and 6.4. Note that, for this coarse grid, ϵ_p grows slightly if n becomes larger. In the computations for Ω_3 (cf. Tables 6.5 and 6.6) we have chosen $i_0 = 3$ as well as $\eta = 1.5$. For smaller i_0, the two-grid method over Ω_3 diverges.

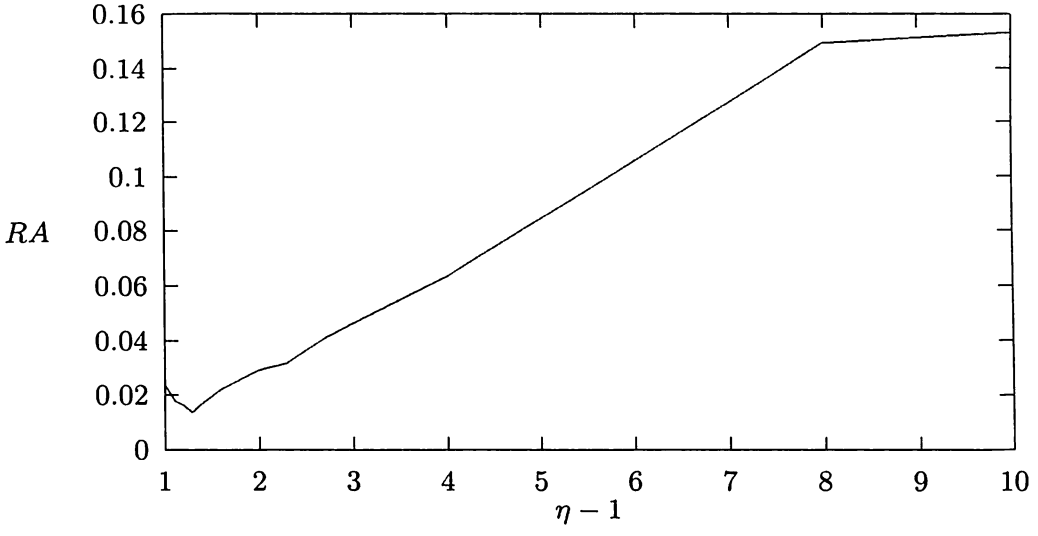

FIG. 6.3. *Convergence factor RA of the two-grid method, no neglect of Str, Ω_2, $i_0 = 1$, $\alpha = \frac{4}{3}$, $n = 6$.*

TABLE 6.3
CPU-times for several solvers, Ω_2, $\alpha = 1$, no neglect of Str.

	GE			NE			TH			TA		
n	4	6	8	4	6	8	4	6	8	4	6	8
N	736	1688	3034	736	1688	3034	736	1688	3034	736	1688	3034
N_c							370	564	758	370	564	758
NI				27	31	35	3	4	5	4	5	6
RA				0.78	0.79	0.79	0.07	0.14	0.19	0.14	0.18	0.21
TI	249	2914	16919	32	185	651	81	351	1147	63	255	726

TABLE 6.4
CPU-times for several solvers, Ω_2, $\alpha = 1$, no neglect of Str.

	GM			TA′			PG			PG′		
n	4	6	8	4	6	8	4	6	8	4	6	8
N	736	1688	3034	736	1688	3034	736	1688	3034	736	1688	3034
N_c				370	564	758	370	564	758	370	564	758
NI	9	10	11	4	5	6	5	6	7	5	6	7
RA	0.37	0.38	0.38	0.14	0.18	0.19	0.09	0.12	0.13	0.09	0.15	0.21
TI	25	141	470	38	162	498	67	274	788	55	213	623

6.3. Comparison of several iteration procedures. We have implemented the following algorithms for the solution of the system of equations (2.4).

TABLE 6.5
CPU-times for several solvers, Ω_3, $\alpha = 1$, no neglect of Str.

	GE			GM			TH			TA		
n	5	7	9	5	7	9	5	7	9	5	7	9
N	720	1344	2160	720	1344	2160	720	1344	2160	720	1344	2160
N_c							696	1104	1488	696	1104	1488
NI				60	75	94	4	5	5	3	3	5
RA				0.99	0.98	0.95	0.22	0.35	0.23	0.17	0.23	0.25
TI	224	1521	6286	111	440	1271	282	1072	2617	245	927	2285

TABLE 6.6
CPU-times for several solvers, Ω_3, $\alpha = 1$, no neglect of Str.

	TA′			PG			PG′		
n	5	7	9	5	7	9	5	7	9
N	720	1344	2160	720	1344	2160	720	1344	2160
N_c	696	1104	1488	696	1104	1488	696	1104	1488
NI	3	3	5	3	3	5	3	3	5
RA	0.17	0.23	0.25	0.12	0.13	0.23	0.12	0.13	0.23
TI	330	984	2303	249	941	2319	389	1325	3556

(a) GE: The first method is the Gaussian elimination (LU-factorization).

(b) GM: The second is the GMRES method (cf. [25]). Before applying this iteration, we multiply both sides of the equation by the inverse of the main diagonal of the matrix. The basis of the Krylov space is obtained by the method of Householder [29], [14], [28] and no restart is performed. A restart would result in slower convergence for Ω_1 and Ω_2 and in no convergence for Ω_3.

(c) TH: We consider the two-grid method of the form (2.8)–(2.11), where the coarse-grid equation (2.10) is solved by GE. Note that this form of the two-grid iteration can be found, for example, in the book by Hackbusch [11].

(d) TA: This is the method (2.8)–(2.11), where the coarse-grid equation is solved by GE and the prolongation defined by Nyström's interpolation is replaced by piecewise constant interpolation. Note that piecewise linear prolongation would not improve the numerical results essentially. The use of Nyström's interpolation for the restriction and piecewise constant interpolation for the prolongation seems to be the most efficient variant of the two-grid iteration since only one multiplication by an $N \times N$-matrix is required. Moreover, we have computed the iterative solutions following the algorithm of Atkinson [4], [5]. Thus the iteration looks as follows: Start with the initial solution $x^0 := 0$ and suppose we are given the iterative solution x^i over the node points of the fine grid. We compute the residual (defect) $r := \frac{1}{2}[1 - W_N(\chi)]^{-1}\{y - A_N x^i\}$ over the node points of the fine grid (multiplication of an $N \times N$-matrix times a vector). Then we determine $p := -2W_N(\chi r)$ over the node points of the coarse grid (multiplication of an $N_c \times N$-matrix times a vector). We solve the coarse-grid equation $A_{N_c} d = p$ to get d over the nodes of the coarse grid. Finally, we compute $x^{i+1} = x^i + r + d'$ over the nodes of the fine grid. Here d' is the piecewise constant interpolation of d, i.e., $d'(P^i) = d(P_c^j)$ if the fine-grid node P^i belongs to the coarse-grid subdomain S_c^j.

(e) TA′: This is the method TA with the only difference that the coarse-grid equation is solved by GM.

(f) PG: We consider the preconditioned GMRES method. Thus instead of solving $A_N x_N = y$ over the nodes of the fine grid, we solve $A^{(-1)} A_N x_N = A^{(-1)} y$, where $A^{(-1)}$

TABLE 6.7
CPU-times for GM and TA, Ω_1, $\alpha = 4/3$, neglect of Str.

		GM			TA		
n	N_n	NI	RA	TI	NI	RA	TI
2	24	2			1		
4	240	3	0.074	2	2	0.024	2
8	1368	5	0.096	72	4	0.162	63
11	2928	6	0.106	353	5	0.219	318

is an approximate matrix for the inverse of A_N. More exactly, $A^{(-1)}$ is the matrix that corresponds to one step of the iteration TA.

(g) PG′: This is the same method as PG with the difference that $A^{(-1)}$ is the matrix that corresponds to one step of the iteration TA′; i.e., the coarse-grid equation in the two-grid preconditioner is solved by GM.

(h) NE: This is the simple iteration (2.12) with $\kappa = 1$ and initial function $x^1 = y/2$.

In Tables 6.3, 6.4, 6.5, and 6.6 we present the computing times TI for the several methods applied to the solution of (2.4) over Ω_2, Ω_3, and Ω_1. The time is given in CPU-seconds. The linear system of equations is solved up to an error of a tenth of the discretization error, and the discretization error is the arithmetic mean of the errors $|x_N(P^i) - x(P^i)|$, $i = 1, \ldots, N$. Furthermore, the coarse-grid equation in PG′ is solved up to an error of 10^{-5} for Ω_3 and 10^{-6} for Ω_2. The coarse-grid solution in TA′ is computed up to 10^{-4} for Ω_3 and up to a tenth of the discretization error for Ω_2. The right-hand side is $y(P) = 6|P - (1,0,0)|^{-2}$ for Ω_2 and $y(P) = 2|P - (1,0,\frac{1}{\sqrt{2}})|^{-1}$ for Ω_3. Note that the number RA for the algorithms GM, PG, and PG′ is the geometric mean value of the ratios $\|x^i - x_N\| / \|x^{i-1} - x_N\|$. For GE, TH, TA, NE, and TA′, the ratio RA is defined as in (6.4). The number of iterations is denoted by NI. From Tables 6.3, 6.4, 6.5, and 6.6 we learn that GM is the most effective solver. However, if n is much higher than in Tables 6.3–6.6, then we expect the methods TA′ and PG′ to be much faster than GM. We have tested higher values of n for the domain Ω_1. For $n = 11$ and $N_n = 2904$, GM requires 402 seconds whereas TA′ only 372. The iteration PG does not seem to be an acceleration of the two-grid iteration though the convergence ratio RA is the best for this algorithm. Numerical tests show that NE diverges for the domain Ω_3.[9] For Ω_2 it is quite fast and better than TA. Furthermore, let us mention that we have obtained similar results for $\alpha = \frac{4}{3}$ and $\alpha = 2$.

In our last numerical example let us consider Ω_1 and neglect the strip *Str*. Let us consider the function (6.3) and choose $\alpha = \frac{4}{3}$, $i_0 = 1$. In this situation $N_c = 24$ is already much smaller than N and the two-grid algorithm should be the fastest. In Table 6.7 we present the CPU-time for the computation including GM and TA. Indeed, it turns out that TA is faster. The number RA for the algorithm GM is the geometric mean value of the ratios $\|x^i - x_N\| / \|x^{i-1} - x_N\|$ and the ratio (6.4) for TA. Thus the convergence factors of the iteration procedures are smaller for TA and, therefore, a smaller number of iteration steps is needed. The method TA can be accelerated by choosing $\eta = 10$ (cf. the introduction of the coarse grid in §5.2). Then, for example, $n = 11$, TA requires 299 seconds only. Note that the time for the computation of the matrices is greater than that for the iteration process provided that n is large and that TA, GM, or TA′ is applied. For instance, 244 seconds of the

[9] The reason for this seems to be that the ϵ_p (cf. (3.3)) for the grids in use are not very small. We conjecture that Theorem 4.2 remains valid even if (3.2) is not fulfilled.

318 for TA and 235 seconds of the 353 for GM (cf. Table 6.7) are needed in order to generate the matrices. All the calculations for the CPU-time (cf. Tables 6.3–6.7) have been performed in double-precision arithmetic on a VAX 4000-300.

Acknowledgment. I am grateful to B. Kleemann for his cooperation in the implementation of the numerical examples and to R. Schlundt for his help during the tests with the GMRES.

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